ON THE PROBLEM OF n BODIES*

BY

J. J. L. HINRICHSEN

Introduction. For the problem of three bodies, Sundman† established together with other results that if the angular momentum of the three bodies is not zero about every axis through the center of gravity of the system, the greatest of the three mutual distances will always exceed a specifiable constant depending upon the initial configuration of the bodies, and hence that triple collision is impossible. The problem was then considered from a different point of view by Birkhoff‡ in his Chicago Colloquium lectures of 1920. He considered the case for which the angular momentum of the three bodies about every axis through the center of gravity of the system is not zero and for which the constant K appearing in the energy integral: $T=U-K$, is (1) equal to or less than zero, and (2) greater than zero. Here T denotes the kinetic energy and $-U$ denotes the potential energy of the system. He showed for the first case that at least two if not all three of the mutual distances increase indefinitely as the time increases and decreases. For the second case, he showed if the motion of the three bodies is such that for some instant all three bodies approach sufficiently near to one another, that two of the mutual distances become infinite with the time while the third mutual distance remains less than a definite constant depending only upon the energy constant and the total mass of the system. After stating and proving various other results, he concluded by stating without formal proof that the results described above may be extended to the case of n bodies attracting one another according to the Newtonian law of force as well as to the case of n bodies attracting one another according to a more general law of force. The present paper has as its object the investigation of the conditions under which these extensions apply.

The equations of motion and other fundamental relationships. We shall denote the n bodies (assumed to be particles) by P_i $(i=1, 2, \cdots, n)$, and suppose them to have positive finite masses m_i and real coordinates (x_i, y_i, z_i). The distance from P_i to P_j will be denoted by r_{ij}. We shall suppose that the bodies attract one another in such a way that there exists a potential function

* Presented to the Society, December 30, 1929; received by the editors July 19, 1932, and, in revised form, July 24, 1933.

† Sundman, *Mémoire sur le problème des trois corps*, Acta Mathematica, vol. 36 (1913), p. 105.

‡ Birkhoff, *Dynamical Systems*, 1927, p. 260. This book is volume IX of the American Mathematical Society Colloquium Publications.

$$U = \sum_{i,j=1}^{n} m_i m_j/(r_{ij})^d, \qquad i \neq j,\ 0 < d < 2.$$

If $d=1$, this function reduces to that for the Newtonian law of attraction. Inasmuch as the probability of collision among particles moving according to this law is zero in the general case, we shall assume that none of the n bodies ever collide. Then all of the r_{ij} will always be positive.

If t denotes the time, the equations of motion will be

$$m_i \frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \qquad m_i \frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \qquad m_i \frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}.$$

The ordinary existence theorems for a system of differential equations may be applied to yield the result that for assigned values of the coordinates and velocity components for $t=\bar{t}$ where $t_0<\bar{t}<t_1$, there exists a unique set of analytic functions $x_i(t)$, $y_i(t)$, $z_i(t)$, $x_i'(t)$, $y_i'(t)$, $z_i'(t)$ defined and satisfying the system of equations for $t_0<t<t_1$ and taking on the assigned values for $t=\bar{t}$. Furthermore, since we assume that the distances r_{ij} are always positive, the interval of definition may be extended to the interval $-\infty<t<\infty$.

The equations of motion admit the following ten integrals:

$$\begin{aligned}
&\sum m_i(x_i'^2 + y_i'^2 + z_i'^2) = 2(U-K),\\
&\sum m_i x_i = \sum m_i y_i = \sum m_i z_i = 0,\\
&\sum m_i x_i' = \sum m_i y_i' = \sum m_i z_i' = 0,\\
&\sum m_i(y_i z_i' - z_i y_i') = c_1,\\
&\sum m_i(z_i x_i' - x_i z_i') = c_2,\\
&\sum m_i(x_i y_i' - y_i x_i') = c_3,
\end{aligned}$$

where the summations for i are to be taken from 1 to n. Here K, c_1, c_2, c_3 are constants of integration and the primes denote derivatives with respect to t. The coordinate system has been so chosen that the center of gravity of the system is fixed at the origin.

If we define

$$R^2(t) = \left(\sum_{i,j=1}^{n} m_i m_j r_{ij}^2 \right)/(2M),$$

where M represents the total mass of the system, it is not difficult to obtain the analogue of Lagrange's Identity*:

$$(R^2)'' = 2(2-d)U - 4K.$$

We shall suppose $0<d<2$ in order that the coefficient of U may be positive.

* Lagrange, *Essai sur le problème des trois corps*, Oeuvres, vol. 6, p. 240.

We shall now proceed to derive the analogue of Sundman's Identity* for the problem of n bodies. Let us choose the coordinate axes in such a way that the products of inertia of the n bodies vanish, and if the moments of inertia about the x, y, z axes are A, B, C respectively, that $A \geqq B \geqq C$. We propose to find a minimum for the kinetic energy of the system

$$T = \tfrac{1}{2} \sum m_i (x_i'^2 + y_i'^2 + z_i'^2),$$

when the $3n$ space coordinates are fixed and the $3n$ velocity components are allowed to vary except for being required to satisfy the integrals of angular momentum and

$$RR' \equiv \sum m_i (x_i x_i' + y_i y_i' + z_i z_i') = c_4.$$

By the Lagrange method of multipliers† the minimum value of T under these conditions is found to be

$$\frac{1}{2}\left(\frac{c_1^2}{A} + \frac{c_2^2}{B} + \frac{c_3^2}{C} + \frac{c_4^2}{R^2}\right) \geqq \frac{1}{2}\left(\frac{f^2}{A} + R'^2\right),$$

where $f^2 = c_1^2 + c_2^2 + c_3^2$. On applying the energy integral, we may express this result by writing

$$R'^2 + P = 2(U - K) \quad \text{where} \quad P \geqq f^2/R^2.$$

Let us now eliminate U between the above two fundamental identities. If we define

$$F \equiv 2RR'' + dR'^2 + 2dK - (2 - d)f^2/R^2,$$

the relationship obtained will show that $F \geqq 0$. Let us define

$$H \equiv R^d(R'^2 + 2K + f^2/R^2),$$

and differentiate with respect to t. In terms of F, we obtain $H' = FR^{d-1} \cdot R'$, from which we have the following result: *If R increases, H cannot decrease, and if R decreases, H cannot increase.* We furthermore note if $f > 0$, R cannot approach zero, since then, by its definition, H would become infinite.

By means of the six integrals of linear momentum, the system of equations of motion may be reduced to a system of order $6n-6$. We shall carry out this reduction in the following manner. For any instant, consider first all possible ways of dividing the n bodies into two groups G_1, G_2, and choose that one for which at the given instant the distance from the center of gravity of one group to that of the complementary group is greatest. There may be

* Sundman, loc. cit., p. 148.

† See for example Goursat, *Cours d'Analyse Mathématique*, 1923, vol. 1, p. 119.

more than one such method of subdivision giving this maximum distance, in which case we shall divide the n bodies into two groups in any one of the several possible ways. Let the coordinates of the center of gravity of one group, G_2, with respect to the center of gravity of the complementary group, G_1, as origin be (ξ_1, η_1, ζ_1) and define $\rho_1^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2$.

If either G_1 or G_2 contains more than one body, consider the various possible ways of subdividing G_1 and G_2 into subgroups, and choose a method of subdividing one group to give the greatest possible distance from the center of gravity of one subgroup to that of the complementary subgroup. Let the coordinates of the center of gravity of one subgroup with respect to the center of gravity of the complementary subgroup as origin be (ξ_2, η_2, ζ_2) and define $\rho_2^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2$. This process of subdivision may be repeated until each of the final groups contains only one body. When this stage has been reached, $n-1$ sets of coordinates (ξ_j, η_j, ζ_j) will have been introduced together with $n-1$ distances defined by $\rho_j^2 = \xi_j^2 + \eta_j^2 + \zeta_j^2$.

The equations of transformation from (x_i, y_i, z_i), $i=1, 2, \cdots, n$, to (ξ_j, η_j, ζ_j), $j=1, 2, \cdots, n-1$, will depend upon the distribution of the n bodies with respect to one another and hence will in general depend upon t. If the position of each of the bodies at a given instant is known, there will always exist at least one way of separating the n bodies into groups in the manner described above, and then the equations of transformation together with their inverse formulas may be written down. If the system of n bodies is divided into groups in a proper manner, the same equations will apply throughout some interval of time containing the given instant. As t increases or decreases, the intervals of time throughout which a particular grouping obtains may become smaller and smaller and approach zero as a limit. Since we exclude the possibility of collision, the velocities of the bodies are bounded and in a sufficiently small interval of time the position of the bodies can change by only a small amount. To make it possible to continue the transformation beyond a limit point of grouping intervals, we shall modify the above method of dividing the bodies into groups in a sufficiently restricted neighborhood of the limit point so as to preserve a constant grouping there. Then by setting up a finite number of sets of equations of transformation, we may for any given finite interval of time express the equations of motion together with the energy integral and the integrals of angular momentum in terms of the new variables (ξ_j, η_j, ζ_j).

Throughout any interval of time $t' < t < t''$ for which $\rho_j(t)$ represents the distance between the centers of gravity of the same two fixed groups of bodies, $\rho_j(t)$ will be analytic. For an instant at which the grouping changes, some distance ρ_j must change to the distance between two new centers of gravity.

In this case all of the following ρ_j will in general also change to distances between new centers of gravity. Except for intervals of time containing limit points of grouping intervals, the ρ_j with the smallest subscript which changes will be continuous but will in general have a discontinuous first derivative. For such intervals of time as contain a limit point of grouping intervals the ρ_j with the smallest subscript which changes will itself have a break whose magnitude may be made arbitrarily small by taking the interval about the limit point small enough. The ρ_j with larger subscripts will in general be discontinuous in either case.

Suppose for $t'<t<t''$, the group of bodies $\sum P_h$ has its center of gravity at the (ξ_j, η_j, ζ_j)-origin and the group $\sum P_k$ has its center of gravity at the point whose coordinates are (ξ_j, η_j, ζ_j). If we define

$$\mu_j = \frac{(\sum m_h)(\sum m_k)}{(\sum m_h) + (\sum m_k)} \qquad (j = 1, 2, \cdots, n-1),$$

the reduced system of equations of order $6n-6$ will assume the simple form

$$\mu_j \frac{d^2\xi_j}{dt^2} = \frac{\partial U}{\partial \xi_j}, \qquad \mu_j \frac{d^2\eta_j}{dt^2} = \frac{\partial U}{\partial \eta_j}, \qquad \mu_j \frac{d^2\zeta_j}{dt^2} = \frac{\partial U}{\partial \zeta_j}.$$

If we denote the derivatives of ξ_j, η_j, ζ_j with respect to t by $\xi_j', \eta_j', \zeta_j'$, the energy integral becomes

$$\sum \mu_j(\xi_j'^2 + \eta_j'^2 + \zeta_j'^2) = 2(U - K),$$

while the integrals of angular momentum become

$$\sum \mu_j(\eta_j\zeta_j' - \zeta_j\eta_j') = c_1,$$
$$\sum \mu_j(\zeta_j\xi_j' - \xi_j\zeta_j') = c_2,$$
$$\sum \mu_j(\xi_j\eta_j' - \eta_j\xi_j') = c_3.$$

Finally, if the values of x_i, y_i, z_i in terms of ξ_j, η_j, ζ_j are substituted in the expression for R^2, we obtain

$$R^2(t) = \tfrac{1}{2} \sum \mu_j \rho_j^2 .$$

Some properties of the motions. We shall now proceed to consider those properties of the motions of the n bodies acting under the above law of force which correspond to the properties considered by Sundman* and Birkhoff† for the problem of three bodies under the Newtonian law of force. With the analogues of the fundamental identities of Lagrange and Sundman together with the (ξ_j, η_j, ζ_j), $j=1, 2, \cdots, n-1$, coordinates available, the proofs of

* Sundman, loc. cit., p. 105.

† Birkhoff, loc. cit., p. 275.

these theorems will be found similar to those for the classical problem. For this reason we shall merely state certain results and outline the proofs of others.

Directly from the analogue of Sundman's Identity we have the following: *For the case* $K \leqq 0$, $0<d<2$, *at least* $n-1$ *of the mutual distances* r_{ij} *increase indefinitely as the time increases or decreases.* We shall now restrict ourselves to the case of $K>0$.

If $K>0$, *the least of the mutual distances* r_{ij} *cannot exceed* $[M^2/(2K)]^{1/d}$. This result follows immediately when the definition of U is applied to the energy integral.

For the case $K>0$, *the largest* r_{ij} *will necessarily exceed* k *times the smallest* r_{ij} *provided*

$$R \leqq \frac{m}{(2M)^{1/2}}\left(\frac{4f^2}{M^3k^2}\right)^{1/(2-d)} \quad or \quad R \geqq \frac{kM^{1/2}}{2}\left(\frac{M^2}{2K}\right)^{1/d},$$

where m *denotes the least of the masses* m_i. Here we must apply the analogue of Sundman's Identity together with inequalities obtained from R.

For the case $K>0$, $0<d<2$, *any part of the curve* $R=R(t)$ $(t, R$ *rectangular coordinates*) *for which* $R<f[(2-d)/(2dK)]^{1/2}$ *consists of a finite arc concave upwards and with a single minimum. If* $R=R_0$ *gives this minimum, the curve rises on either side until* R *satisfies the inequality*

$$(R^d - R_0{}^d)/[1-(R_0/R)^{2-d}] \geqq f^2/(2KR_0^{2-d}),$$

with a corresponding slope R' *at least as great as is demanded by the inequality*

$$R'^2 \geqq E^2 \equiv f^2\left(\frac{1}{R^dR_0^{2-d}} - \frac{1}{R^2}\right) + 2K\left(\frac{R_0{}^d - R^d}{R^d}\right),$$

at every intermediate stage. This result follows from a combination of the analogue of Lagrange's and the analogue of Sundman's Identity.

Since we are considering motions for which there are no collisions, f must be positive before this theorem may be applied. The n bodies are all near together at some instant $t=t_0$, the amount of separation being measured by R. The bodies separate in such a way that R increases and very rapidly as long as R is not too small or large until R has become very large. Since the least of the mutual distances is not greater than $[M^2/(2k)]^{1/d}$ for all values of the time, at least two bodies must remain relatively near together throughout the entire motion.

We shall now turn to consider the function $\rho_1(t)$. We shall prove the following theorem:

In the case $K>0$, throughout any interval of time $\rho_1'' > -3dM^{d+2}/(2m\rho_1)^{d+1}$. If for any instant $\rho_1' > [3M^{d+2}/(2^d m^{d+1}\rho_1^d)]^{1/2}$, then ρ_1 will continue to increase indefinitely with t.

Let us first consider t in an interval $t'<t<t''$ sufficiently restricted so that the n bodies preserve one and the same grouping throughout. If one group consists of the k bodies $P_i (i=1, 2, \cdots, k; k=1, 2, \cdots, n-1)$, while the complementary group consists of the $n-k$ bodies P_j, $j=k+1, \cdots, n$, then we can show that there exists a positive lower bound for the distances r_{ij} in terms of ρ_1, namely $r_{ij} \geqq 2m\rho_1/M$ for $i=1, 2, \cdots, k; j=k+1, \cdots, n$.

The distance ρ_1 may be written $\rho_1 = MW/\mu^2$, where

$$\mu^2 = \sum_{j=k+1}^{n} \sum_{i=1}^{k} m_i m_j \text{ and } W^2 = \left(\sum_{i=1}^{k} m_i x_i\right)^2 + \left(\sum_{i=1}^{k} m_i y_i\right)^2 + \left(\sum_{i=1}^{k} m_i z_i\right)^2.$$

Upon differentiating twice with respect to t and dropping three non-negative terms from the second member, we obtain

$$\rho_1'' \geqq [M/(\mu^2 W)][(\sum m_i x_i)(\sum m_i x_i'') + (\sum m_i y_i)(\sum m_i y_i'') + (\sum m_i z_i)(\sum m_i z_i'')],$$

where the summations are to be taken from $i=1$ to $i=k$. If furthermore we use the equations of motion to eliminate the second-order derivatives and simplify by applying inequalities of the type

$$x_i - x_j \leqq r_{ij}, \qquad \sum_{i=1}^{k} m_i x_i \leqq W$$

we obtain the desired inequality concerning ρ_1''. By integrating both sides of this inequality, we find if for any t in $t'<t<t''$ the inequality involving ρ_1' is satisfied, that ρ_1 will continue to increase indefinitely if the grouping of the n bodies does not change. For any instant that the grouping does change, either ρ_1' will be continuous or it will be increased and hence if this inequality is satisfied for any instant, ρ_1 will continue to increase indefinitely with t.

We proceed to combine these results in order to show that a motion having its minimum R sufficiently small is one for which R and ρ_1 increase indefinitely as t increases or decreases. According to what has been proved, for R^* and $R^{*\prime}$ arbitrarily large and for any fixed d, $0<d<2$, a positive R_0 can be chosen so small that all motions for which the minimum R is not more than R_0 correspond to an R which increases from the minimum to R^* and has for $R=R^*$ a derivative R' which is at least as great as $R^{*\prime}$.

The function $\rho_1(t)$ is defined throughout any finite interval of time and will satisfy the inequality

$$2R/[(n-1)M^{1/2}] < \rho_1 < (2M)^{1/2}R/m,$$

from which it is evident that if R increases indefinitely so also must ρ_1, and conversely if ρ_1 increases indefinitely so also must R.

Let us consider a fixed value of R_0 satisfying the inequality

(a) $$0 < R_0 < (2-d)^{1/2}f/(2dK)^{1/2}.$$

Then R must increase until

$$R^d/[1-(R_0/R)^{2-d}] \geqq f^2/(2KR_0^{2-d}).$$

Given any value R^*, we can choose R_0 so small that R becomes greater than R^*. We shall suppose therefore that R_0 has been chosen so small that in addition to satisfying (a) the motion is such that R increases until $R \geqq 2^{1/(2-d)}R_0$. In this case R increases from R_0 until $2R^d \geqq f^2/(2KR_0^{2-d})$ or until $R \geqq R^* \equiv f^{2/d}/(2^2KR_0^{2-d})^{1/d}$. The above inequality will be satisfied if R_0 is chosen so small that $f^{2/d}/(2^2KR_0^{2-d})^{1/d} \geqq 2^{1/(2-d)}R_0$, or if

(b) $$R_0 \leqq f/(2^{(4-d)/(2-d)}K)^{1/2}.$$

Now let us define $R^{**} = mR^*/[2^{3/2}(n-1)M]$. If we choose R_0 so small that

(c) $$R_0 < m^{d/2}f/[2^{(5d+4)/2}(n-1)^dM^dK]^{1/2},$$

we shall have $R_0 < R^{**}/2$. If we define $R^{***} = R^*/2$, it is obvious that $R^{**} < R^{***}$. If t_0 denotes the first value of t for which $R(t) = R_0$, and t^* denotes the first value of t greater than t_0 for which $R(t) = R^*$, there will exist a unique pair of values t^{**}, t^{***} in the interval $t_0 < t < t^*$ such that $R(t^{**}) = R^{**}$ and $R(t^{***}) = R^{***}$. Since the function $R(t)$ is continuous and has a continuous derivative for t in any closed interval $t^{**} \leqq t \leqq t^{***}$, we may apply the law of the mean for derivatives which states that there exists at least one point $\bar{t}$ in $t^{**} \leqq t \leqq t^{***}$ such that $(R^{***}-R^{**})/(t^{***}-t^{**}) = \overline{R}'$ where $\overline{R}' = R'(\bar{t})$. Since $\overline{R} < R^*$, $\overline{R}'$ must satisfy our previous inequality and $(R^{***}-R^{**})/(t^{***}-t^{**}) \geqq \overline{E}$ where $\overline{E}$ denotes E with R replaced by $\overline{R}$.

Consider now the average rate of change of ρ_1 throughout the interval $t^{**} \leqq t \leqq t^{***}$. We find

$$[\rho_1(t^{***}) - \rho_1(t^{**})]/[t^{***} - t^{**}] > [R^{***} - R^{**}]/[(n-1)M^{1/2}(t^{***} - t^{**})].$$

There must exist a value of t, say $\bar{t}_1$, satisfying $t^{**} < \bar{t}_1 < t^{***}$, such that $\rho_1'(\bar{t}_1) \geqq \overline{E}/[(n-1)M^{1/2}]$.

We wish to show that a motion having its minimum R denoted by R_0 small enough is one for which R and ρ_1 become infinite with t. This result will follow if $\overline{E}/[(n-1)M^{1/2}] > [(2^{2-d}M^{2+d})/(m^{d+1}\rho_1^d(\bar{t}_1))]^{1/2}$, or on eliminating ρ_1 if

$$(m^{2d+1}f^2\overline{E}^2)/\left[2^{(8-d)/2}(n-1)^{2(d+1)}M^{(5d+6)/2}KR_0^{2-d}\right] > 1.$$

It is obvious by the choice of R^{**} and R^{***}, that $[R_0^d-\overline{R}^d]/\overline{R}^d>-1$. Since $\overline{R}<R^*/2$, we have $1/(\overline{R}^dR_0^{2-d})>2^{2+d}K/f^2$. Also since $\overline{R}>R^{**}$ we have

$$-1/\overline{R}^2 > -2^{(3d+4)/d}(n-1)^2M^2K^{2/d}R_0^{2(2-d)/d}/\left[m^2f^{4/d}\right].$$

If furthermore we suppose

$$\text{(d)} \qquad R_0 < \frac{m^{d/(2-d)}f}{2^{(4+2d-d^2)/[2(2-d)]}(n-1)^{d/(2-d)}M^{d/(2-d)}K^{1/2}},$$

then

$$f^2\left(\frac{1}{\overline{R}^dR_0^{2-d}}-\frac{1}{\overline{R}^2}\right) > 2^{d+1}K,$$

and the desired inequality will be satisfied if

$$\text{(e)} \qquad R_0 < \frac{(2^d-1)^{1/(2-d)}m^{(2d+1)/(2-d)}f^{2/(2-d)}}{2^{(6-d)/[2(2-d)]}(n-1)^{2(d+1)/(2-d)}M^{(5d+6)/[2(2-d)]}}.$$

We have the following result: *In the case* $K>0$, $0<d<2$, *if the motion is such that the n bodies approach so closely that the minimum R denoted by* R_0 *satisfies the inequalities* (a), (b), (c), (d) *and* (e), *then at least* $n-1$ *mutual distances become infinite with t while at least one such distance remains less than* $[M^2/(2K)]^{1/d}$.

We may also state one further property of motions of the above kind. *Any motion for which* $f>0$, $K>0$, $0<d<2$ *and the bodies are all near together at some instant* $t=t_0$ *is characterized by the property that one* r_{ij} *remains relatively large compared to the smallest* r_{ij} *throughout the entire motion.* This result follows from an earlier result, the definition of R, the energy integral and the analogue of Sundman's Identity.

The results of this paper may be extended to motions embracing instants of collision if any kind of continuation after multiple collision were possible in which the constants of linear and angular momentum as well as of energy are the same after as before collision and if also R' may be regarded as continuous at collision. In this case none of the analytic work would be affected even though for certain instants there did occur multiple collisions among the bodies.

IOWA STATE COLLEGE,
AMES, IOWA